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Zamolodchikov–Faddeev algebra in two-component anyons

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Abstract. We investigate Wilczeck's mutual fractional statistical model at the field-theoretical level. The effective Hamiltonian for the particles is derived by the canonical procedure, whereas the commutators of the anyonic excitations are proved to obey the Zamolodchikov–Faddeev algebra. Cases leading to well known statistics as well as Laughlin's wavefunction are discussed.

Fractional statistics plays an important role in planar physics. In two-dimensional space identical particles can obey new kinds of statistics [1], interpolating between the normal Bose and Fermi statistics. The general theory of fractional statistics was put forward by Wilczeck and Zee [2] and Wu [3] who interpreted the theory in the path integral formalism and related the theory to the braid group, while the connection with the Chern–Simons theory was also proposed [4, 5]. Since then, a new subject, called anyonic physics, has been widely developed, especially in the application to high-T_c superconductivity [6] and the quantum Hall effect [7].

Recently Wilczeck has proposed a two-component anyonic model to describe incompressible liquid quantized Hall states in situations where two distinct kinds of electrons are relevant. The model can also overcome the violation of the discrete symmetries P and T in the original anyonic superconductivity theorem. In his paper [8] mutual statistics between distinguished particles are introduced, which brings about some interesting results.

Multi-anyon quantum mechanics poses a challenging problem in theoretical physics. A complete solution is far out of reach, and we can only understand some features from a different point of view. One of them is to know the statistical character of the anyonic excitation. In this work, based on the second quantized procedure, we diagonalize the Hamiltonian for the two-component anyons, at the expense of the unusual commutation relation of the field operators, which shares the Zamolodchikov–Faddeev algebra. Though the method here is not developed for the first time, the result has never appeared in the literature. The algebra of these operators reflects the statistical characters between the anyonic excitations. Special values for the coupling constants leading to the Bose, Fermi and q-deformed statistics are discussed and, meanwhile, the Laughlin's wavefunction is given.

Let us start from Wilczeck's model [8],

$$L = \sum_{i,a=1,2} \left\{ \frac{1}{2} M \left(\frac{\mathrm{d}\boldsymbol{x}_i^{(a)}}{\mathrm{d}t} \right)^2 + \frac{e}{c} a^{(a)}(\boldsymbol{x}_i) \frac{\mathrm{d}\boldsymbol{x}_i^{(a)}}{\mathrm{d}t} - e a_0^{(a)}(\boldsymbol{x}_i) \right\} + \frac{1}{2\pi} \int \mathrm{d}\boldsymbol{x}\, n_{ab} \epsilon^{\alpha\beta\gamma} a_\alpha^{(a)} f_{\beta\gamma}^{(b)} \tag{1}$$

where

$$f^{(a)}_{\beta\gamma} = \partial_\beta a^{(a)}_\gamma - \partial_\gamma a^{(a)}_\beta .$$

In equation (1) the index i refers to a given particle, α, β, γ correspond to the three-dimensional Lorentz coordinates, (a) represents the two kinds of particles, and all coordinates $\boldsymbol{x}$ or $\boldsymbol{x}_i$ refers to two-dimensional vectors. The matrix

$$n_{ab} = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix}$$

describes the mutual statistics between the two kinds of particles and their self-statistics.

Since the time components of the gauge fields are Lagrangian multiples, one proceeds by eliminating them by varying the Lagrangian with respect to them ($\delta L/\delta a^0 = 0$). Then one gets

$$e \sum_i \delta(\boldsymbol{x} - \boldsymbol{x}^{(a)}_i) = \sum_{b=1,2} \frac{n_{ab}}{\pi} (\partial_1 a^{(b)}_2 - \partial_2 a^{(b)}_1). \tag{2}$$

Due to the two-dimensional identity $\nabla(\boldsymbol{x}/\boldsymbol{x}^2) = 2\pi\delta(\boldsymbol{x})$, and choosing the Coulomb gauge $\nabla a(\boldsymbol{x}) = 0$, we can suppose the solution of (2) as

$$\begin{aligned} a^{(1)}(\boldsymbol{x}) &= \sum_i \frac{\theta_{11}}{2\pi} \frac{\boldsymbol{k} \times (\boldsymbol{x} - \boldsymbol{x}^{(1)}_i)}{(\boldsymbol{x} - \boldsymbol{x}^{(1)}_i)^2} + \sum_i \frac{\theta_{12}}{2\pi} \frac{\boldsymbol{k} \times (\boldsymbol{x} - \boldsymbol{x}^{(2)}_i)}{(\boldsymbol{x} - \boldsymbol{x}^{(2)}_i)^2} \\ a^{(2)}(\boldsymbol{x}) &= \sum_i \frac{\theta_{21}}{2\pi} \frac{\boldsymbol{k} \times (\boldsymbol{x} - \boldsymbol{x}^{(1)}_i)}{(\boldsymbol{x} - \boldsymbol{x}^{(1)}_i)^2} + \sum_i \frac{\theta_{22}}{2\pi} \frac{\boldsymbol{k} \times (\boldsymbol{x} - \boldsymbol{x}^{(2)}_i)}{(\boldsymbol{x} - \boldsymbol{x}^{(2)}_i)^2} \end{aligned} \tag{3}$$

where $\boldsymbol{k}$ is the unit vector perpendicular to the plane.

The assumption turns out to be true when the parameters θ_{ij} satisfy

$$\begin{aligned} e\pi &= m_1\theta_{11} + n\theta_{21} \\ e\pi &= m_2\theta_{22} + n\theta_{12} \\ 0 &= m_1\theta_{12} + n\theta_{22} \\ 0 &= m_2\theta_{21} + n\theta_{11} \end{aligned} \tag{4}$$

whose solution reads

$$\begin{aligned} \theta_{11} &= \frac{m_2 e\pi}{m_1 m_2 - n^2} \\ \theta_{12} &= \frac{-ne\pi}{m_1 m_2 - n^2} \\ \theta_{21} &= \frac{-ne\pi}{m_1 m_2 - n^2} \\ \theta_{22} &= \frac{m_1 e\pi}{m_1 m_2 - n^2}. \end{aligned} \tag{5}$$

After the second quantization the effective Hamiltonian for the particles reads

$$H = \sum_a \int \mathrm{d}\boldsymbol{x} \frac{1}{2M} \left[\left(-\mathrm{i}\nabla - \frac{e}{c} a^{(a)}(\boldsymbol{x})\right) \Psi^{(a)}(\boldsymbol{x})\right]^\dagger \left[\left(-\mathrm{i}\nabla - \frac{e}{c} a^{(a)}(\boldsymbol{x})\right) \Psi^{(a)}(\boldsymbol{x})\right] \tag{6}$$

where

$$a^{(a)}(\boldsymbol{x}) = \sum_{b=1,2} \frac{\theta_{ab}}{2\pi} \nabla \int \mathrm{d}\boldsymbol{y}\, \theta(\boldsymbol{x} - \boldsymbol{y}) \Psi^{(b)\dagger}(\boldsymbol{y}) \Psi^{(b)}(\boldsymbol{y}) \tag{7}$$

and $\theta(\boldsymbol{x}-\boldsymbol{y})$ is the azimuthal angle of the vector from $\boldsymbol{y}$ to $\boldsymbol{x}$ which is usually a multivalued function. Since we are only interested in the effect of interchanging a pair of particles once, [9], to see the statistics, the first sheet of the complex plane is concerned here, information from higher ones can be trivially derived [10, 11].

Using the Jordan–Wigner transformation [9, 10, 12] we can turn the interacted Hamiltonian into a free one being accompanied by the complicated field operators, or anyonic operators. A study of these operators reveals the statistics of the anyonic excitation mode. The transformation

$$\Psi_F^{(a)}(\boldsymbol{x}) = \exp\left(- \mathrm{i} \sum_b \frac{\theta_{ab}}{2\pi} \int \mathrm{d}z\, \theta(\boldsymbol{x}-\boldsymbol{z}) \Psi^{(b)\dagger}(\boldsymbol{z}) \Psi^{(b)}(\boldsymbol{z}) \right) \Psi^{(a)}(\boldsymbol{x}) \tag{8}$$

makes the Hamiltonian (6)

$$H_F = \sum_a \int \mathrm{d}\boldsymbol{x} \frac{1}{2M} (-\mathrm{i}\nabla \Psi_F^{(a)}(\boldsymbol{x}))^\dagger (-\mathrm{i}\nabla \Psi_F^{(a)}(\boldsymbol{x})). \tag{9}$$

As a note, the free Hamiltonian does not mean a non-interacted system, because we will deal with a complicated commutation algebra.

By a lengthy, but not difficult, calculation we can obtain the commutators of the anyonic operators. For instance, from (8) and using the Baker–Cambell–Hausdorff formula, we obtain

$$\Psi_F^{(1)}(\boldsymbol{x}) \Psi_F^{(1)\dagger}(\boldsymbol{y}) = \delta_{\boldsymbol{x}\boldsymbol{y}} - \mathrm{e}^{\mathrm{i}(\theta_{11}/2\pi)(\theta(\boldsymbol{y}-\boldsymbol{x})-\theta(\boldsymbol{x}-\boldsymbol{y}))} \Psi_F^{(1)\dagger}(\boldsymbol{y}) \Psi_F^{(1)}(\boldsymbol{x}). \tag{10}$$

For simplicity, we define the ordering in two-dimensional space. Given two vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and their components $x_{1,2}$, $y_{1,2}$, we define

$$\begin{aligned} \boldsymbol{x} > \boldsymbol{y} \qquad & \text{if } x_2 < y_2 \\ & \text{or } x_2 = y_2, x_1 < y_1 \\ \boldsymbol{x} = \boldsymbol{y} \qquad & \text{if } x_1 = y_1, x_2 = y_2. \end{aligned} \tag{11}$$

Further, we define

$$\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y}) = \begin{cases} 1 & \boldsymbol{x} > \boldsymbol{y} \\ 0 & \boldsymbol{x} = \boldsymbol{y} \\ -1 & \boldsymbol{x} < \boldsymbol{y}. \end{cases} \tag{12}$$

With these definitions we get

$$\Psi_F^{(1)}(\boldsymbol{x}) \Psi_F^{(1)\dagger}(\boldsymbol{y}) = \delta_{\boldsymbol{x}\boldsymbol{y}} - \mathrm{e}^{\mathrm{i}(\theta_{11}/2)\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})} \Psi_F^{(1)\dagger}(\boldsymbol{y}) \Psi_F^{(1)}(\boldsymbol{x}). \tag{13}$$

The same procedures for other commutators lead to the following relation in a compact form,

$$\begin{aligned} \Psi_F^{(i)}(\boldsymbol{x}) \Psi_F^{(j)}(\boldsymbol{y}) &= S_{kl}^{ij}(\boldsymbol{x}-\boldsymbol{y}) \Psi_F^{(k)}(\boldsymbol{y}) \Psi_F^{(l)}(\boldsymbol{x}) \\ \Psi_F^{(i)\dagger}(\boldsymbol{x}) \Psi_F^{(j)\dagger}(\boldsymbol{y}) &= S_{lk}^{*ji}(\boldsymbol{y}-\boldsymbol{x}) \Psi_F^{(k)\dagger}(\boldsymbol{y}) \Psi_F^{(l)\dagger}(\boldsymbol{x}) \\ \Psi_F^{(i)}(\boldsymbol{x}) \Psi_F^{(j)\dagger}(\boldsymbol{y}) &= S_{lj}^{ki}(\boldsymbol{y}-\boldsymbol{x}) \Psi_F^{(k)\dagger}(\boldsymbol{y}) \Psi_F^{(l)}(\boldsymbol{x}) + \delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \end{aligned} \tag{14}$$

where S_{kl}^{ij} is a 4×4 matrix, whose non-zero elements are

$$\begin{aligned} S_{11}^{11}(\boldsymbol{x}-\boldsymbol{y}) &= -\mathrm{e}^{\mathrm{i}(\theta_{11}/2)\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})} \\ S_{12}^{21}(\boldsymbol{x}-\boldsymbol{y}) &= -\mathrm{e}^{\mathrm{i}(\theta_{12}/2)\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})} \\ S_{21}^{12}(\boldsymbol{x}-\boldsymbol{y}) &= -\mathrm{e}^{\mathrm{i}(\theta_{21}/2)\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})} \\ S_{22}^{22}(\boldsymbol{x}-\boldsymbol{y}) &= -\mathrm{e}^{\mathrm{i}(\theta_{22}/2)\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})}. \end{aligned} \tag{15}$$

It is easy to show that the matrix S_{kl}^{ij} satisfies the Yang–Baxter equation:

$$S_{kl}^{ij}(\boldsymbol{x}-\boldsymbol{y})S_{np}^{lm}(\boldsymbol{z}-\boldsymbol{y})S_{qr}^{kn}(\boldsymbol{z}-\boldsymbol{x}) = S_{kl}^{jm}(\boldsymbol{z}-\boldsymbol{x})S_{qn}^{ik}(\boldsymbol{z}-\boldsymbol{y})S_{rp}^{nl}(\boldsymbol{x}-\boldsymbol{y}). \tag{16}$$

The algebras (14) and (16) originally appeared in the $(1+1)$-dimensional integrable field theory [13–15], but it comes into the $(2+1)$-dimensional system, and the spectral parameter in the quantum inverse scattering method is replaced here by the spacelike vector. It is not clear how to interpret the physical consequences of the algebra, because our system is far from the integrable system, but from it we can find some simple and solvable cases.

(1) We have two kinds of free fermions if

$$\theta_{ij} = 4\pi \times \text{integer}. \tag{17}$$

(2) One kind is a free fermion, the other a q-deformed excitation if

$$\theta_{12} = \theta_{21} = 4\pi \times \text{integer} \qquad \theta_{11} = 4\pi \times \text{integer} \qquad \theta_{22} \neq 4\pi \times \text{integer}. \tag{18}$$

(3) We have two kinds of q-deformed excitations, anticommutated to each other, if

$$\theta_{12} = \theta_{21} = 4\pi \times \text{integer} \qquad \theta_{11} \neq 4\pi \times \text{integer} \qquad \theta_{22} \neq 4\pi \times \text{integer}. \tag{19}$$

(4) We have no free excitation mode if

$$\theta_{12} \neq 4\pi \times \text{ integer}. \tag{20}$$

Correspondingly, if we take $\theta_{ij} = 2\pi(2k+1)$, $k \in N$, we have the boson or q-deformed excitation mode.

Finally, introducing two sets of complex coordinates $(z_i, \bar{z}_i)$ and $(w_i, \bar{w}_i)$ for the two kinds of particles, we can discuss the problem in the first quantized formalism. In parallel to (8) and (9), we have a Hamiltonian without interaction

$$H_F' = \sum_{a=1,2} \frac{P_{(a)}^2}{2M}. \tag{21}$$

Correspondingly the wavefunction is multivalued. We can construct Laughlin's wavefunction [7]

$$|\Psi\rangle_F' = \prod (z_i - z_j)^{\theta_{11}/\pi}(w_i - w_j)^{\theta_{22}/\pi}(z_i - w_j)^{\theta_{12}/\pi} f(z_i, \bar{z}_i; w_i, \bar{w}_i) \tag{22}$$

where $f(z_i, \bar{z}_i; w_i, \bar{w}_i)$ is invariant under the interchanging between any two particles.

The following remarks are in order.

(1) If we consider the non-zero winding numbers of the field operators, the coefficients of the exchange algebra will be changed, while the whole algebra keeps the same structure. Actually, in this case, we have an ordered algebra, with the result in this paper as the lowest sub-algebra.

(2) From the free Hamiltonian (9) and the complicated exchange algebra (14), we can derive the Heisenberg equation for these field operators; however, it does not provide us with further information about this system. A possible approach to quantize these braiding fields has been suggested by Bożejko and Speicher [16], but it is still far from the physical requirement.

(3) As an interesting result, we find that the statistical phases between the particles are exactly related to the inverse matrix of the matrix of the statistical interaction. This has also been pointed out in Wilczeck's original paper [8], when he discussed the quasiholes.

(4) When we discuss the problem in the first quantization form, the particles look free, but actually the interaction has entered into the statistical phase θ_{ij}. In this description, the momentum operators have the usual meaning.

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